Multi-monopoles and Magnetic Bags

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Abstract

By analogy with the multi-vortices, we show that also multi-monopoles become magnetic bags in the large n limit. This simplification allows us to compute the spectrum and the profile functions by requiring the minimization of the energy of the bag. We consider in detail the case of the magnetic bag in the limit of vanishing potential and we find that it saturates the Bogomol'nyi bound and there is an infinite set of different shapes of allowed bags. This is consistent with the existence of a moduli space of solutions for the BPS multi-monopoles. We discuss the string theory interpretation of our result and also the relation between the 't Hooft large n limit of certain supersymmetric gauge theories and the large n limit of multi-monopoles. We then consider multi-monopoles in the cosmological context and provide a mechanism that could lead to their production.

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1 Introduction

In a recent series of works [1, 2, 3] we studied the behavior of the Abrikosov-Nielsen-Olesen (ANO) multi-vortex [4, 5] in the large n limit, where n is the number of quanta of magnetic flux. In this limit the multi-vortex becomes a wall vortex, that is a wall compactified on a cylinder and stabilized by the magnetic flux inside. The wall vortex is essentially a bag, such as the ones studied in the context of the bag models of hadrons [7, 6, 8]. When we say *bag*, we mean any object that is composed by a wall of tension $T_{\rm W}$ and thickness $\Delta_{\rm W}$, that separates an internal region with energy density ε_0 and an external region with energy density 0. There are two kind of forces that comprise the bag: one comes from the tension of the wall $T_{\rm W}$ and the other comes from the internal energy density ε_0. If the first dominates we use the name SLAC bag [6], while if the second dominates we use the name MIT bag [7]. The bag, to be stable, needs also some pressure force to balance the collapse forces. In the case of the wall vortex this force is due to the magnetic flux inside the cylinder.

In the present paper we consider the 't Hooft-Polyakov magnetic monopole [9] and we will see that also in this case the multi-monopole becomes a bag in the large n limit. This object, which we from now on will denote *magnetic bag*, is composed by a domain wall of thickness $1/M_W$ (where M_W is the mass of the $W^\pm$ bosons) wrapped around a closed surface $\mathcal{S}_{\rm m}$. The surface $\mathcal{S}_{\rm m}$ can be spherical or can also have different shapes. The shape of the surface encodes in some sense the information about the point on the moduli space of the multi-monopole. The size of the bag is of order n/M_W so in the large n limit it is much bigger than the thickness of the wall. In what follows we will consider the simplest model that admits magnetic monopoles: the $SU(2)$ gauge theory spontaneously broken to $U(1)$ by an adjoint scalar field. The bag surface $\mathcal{S}_{\rm m}$ separates two regions, the internal one that is in the non-abelian $SU(2)$ phase and the external one that is in the abelian phase. The size of the bag is determined by the balance of the attractive and repulsive forces. Attractive forces are due to the internal energy density, the tension of the wall and, in case of vanishing potential, the long tail of the Higgs field outside the bag. The repulsive force is instead due to the abelian magnetic field outside the bag.

It is important to say that this paper does not give a rigorous proof that multi-monopoles become magnetic bags in the large n limit, but only some evidence for it. For this reason we will sometime refer to our claim as the *magnetic bag conjecture*. Let us make a parallel with the multi-vortices. In this case we started in [1, 2]

giving some evidences for the fact that in the large n limit multi-vortices become wall vortices. There where two kind of evidence: a qualitative analysis of the multi-vortex differential equations and the surprising coincidence that the wall vortex saturates exactly the BPS bound. At this level the idea was only a conjecture but in [3] we provided an accurate numerical analysis of the differential equations that, in our opinion, gave a quite convincing proof of our assertion. In the present paper we will follow a similar path for the magnetic bag conjecture. First we will give a motivation from the qualitative behavior of the Higgs field profile at the origin (polynomial $\propto r^n$) and at infinity (it approaches 1 exponentially as $\propto e^{-M_W r}$). Then we will find that the magnetic bag saturates exactly the BPS bound. We will also find that there is an infinite set of different shapes allowed for the magnetic bag and this matches with the existence of a moduli space of solutions for the multi-monopole. The last step is to check the conjecture on some large n solution. One major difference with respect to the multi-vortex is that there is not a spherically symmetric multi-monopole with charge greater than one [12]. The maximal possible symmetry is the axial one that has been studied in great detail in Refs. [13, 15] and there are exact formulas for the profile functions of the multi-monopoles. Using these solutions we can find a quite convincing evidence for the magnetic bag conjecture in the case of the axial symmetric multi-monopole.

In this paper we will also interpret the magnetic bag in the string theory framework. Magnetic monopoles can be realized in string theory as D1-branes suspended between D3-branes. A D1-brane ending on a single D3-brane can be viewed as a deformation of the world volume of the D3-brane. This deformation, also called BIon, is a spike with profile $\propto 1/r$ emerging from the D3-brane and is a solution of the abelian Dirac-Born-Infeld (DBI) action that describe the low energy degrees of freedom on the D3-brane [33]. When considering the D1-brane suspended between two D3-branes we have to use the non-abelian generalization of the DBI action and the D1-brane becomes a couple of spikes emerging from the two D3-branes and ending in a common point [34]. The profile of the two spikes is the same as that of the Higgs field solution in the BPS 't Hooft-Polyakov monopole. When we take a large number of D1-branes at the same position, the point where the two spikes are joined together expand out in a closed surface which is nothing but the boundary of the magnetic bag. Inside the bag are the two D3-branes, one on the top of the other and the non-abelian $SU(2)$ symmetry is restored. Outside of the bag the two D3-branes are deformed in the same way as a cut abelian BIonic spike. The "non-abelianity" survives only in a

thin region of thickness $1/M_W$ around the surface of the magnetic bag.

In the last part of the paper we want to explore the possible formation of multi-monopoles in the cosmological context. Magnetic monopoles are naturally part of the spectrum of grand unified theories (GUTs). Whenever the grand unification group is semi-simple, there are 't Hooft-Polyakov monopoles with typical mass of the GUT scale 10^{15} GeV. The only way to produce these super-heavy defects, is in the cosmological context when the temperature of the universe was of order of the GUT scale. When the temperature cooled below the critical GUT temperature, the GUT Higgs boson condensed in causally disconnected domains. At the intersection of different domains there is some probability p (~ 0.1) to find a monopole of charge 1. The probability to find a multi-monopole at this stage can be neglected. We will find that for a particular choice of the GUT Higgs potential multi-monopoles can be produced after the phase transition and, with an extreme choice of parameters, our mechanism provides also a new possible solution of the cosmological monopole problem.

The paper is organized as follows. In Section 2 we give a brief review of the 't Hooft-Polyakov monopole. This part contains no original material, it is just a collection of results that we will use in the rest of the paper. In Section 3 we discuss multi-monopoles and magnetic bags in the large n limit. In Section 4 we study the moduli space of BPS bags. In Section 5 we clarify some aspects of the magnetic bag conjecture and we find more evidence for it studying the exact known solution of the axial symmetric multi-monopoles. In Section 7 the relation between the 't Hooft large n limit of certain supersymmetric gauge theories and the large n limit of multi-monopoles. In Section 6 we interpret our results in the context of string theory. We conclude in Section 8 speculating about the possible production of GUT multi-monopoles in the cosmological context.

2 Magnetic Monopoles

We consider the simplest unified theory that admits magnetic monopoles (see [57, 58] for reviews). It is as $SU(2)$ gauge theory coupled to a scalar field in the adjoint representation

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - V(\phi) \ , \qquad (2.1)$$

where the covariant derivative is $D_\mu \phi^a = \partial_\mu \phi^a + e e^{abc} A^b_\mu \phi^c$ and e is the coupling constant. The potential is

$$V(\phi) = \frac{1}{8}\lambda(\phi^a\phi^a - v^2)^2 \ . \tag{2.2}$$

The potential (shown in Figure 1) is such that the vacuum manifold is the sphere $\mathbf{S}^2$ of the vectors of fixed norm $|\phi| = v$. The scalar field condensate breaks the gauge

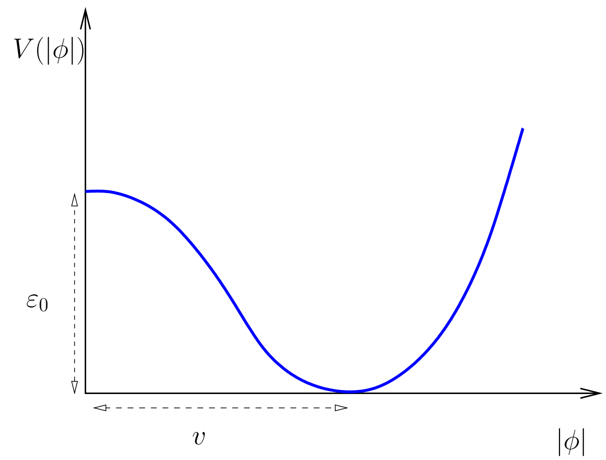

Figure 1: The Higgs potential

group down to $U(1)$. Expanding around the vacuum we obtain the masses of the perturbative particles, respectively the $W^\pm$ bosons and the Higgs boson:

$$M_W = ev \ , \qquad M_H = \sqrt{\lambda} v \ . \tag{2.3}$$

Due to the non trivial topology of the vacuum manifold, the theory admits a nonperturbative particle: the 't Hooft-Polyakov magnetic monopole [9]. It is constructed in the following way. The second homotopy group of the vacuum manifold is $\pi_2(\mathbf{S}^2) = \mathbb{Z}$ and so we can choose a non-trivial configuration such that the spatial sphere $\mathbf{S}^2$ at infinity is winded once around the vacuum manifold $\mathbf{S}^2$. In order to have a finite energy configuration, we have to choose the gauge field such that the covariant derivative $D_\mu \phi$ vanishes rapidly enough at infinity. The monopole field distribution, in spherical coordinates, is

$$\phi^a = v \hat{r}^a H(M_W r) \ , \qquad A^a_i = \frac{\epsilon_{iak}\hat{r}^k}{er}(1 - K(M_W r)) \ , \tag{2.4}$$

where the profile functions must satisfy the boundary conditions $H(\infty) = 1$, $K(\infty) = 0$ and $H(0) = 0$, $K(0) = 1$. The profiles H and K are functions of a dimensionless variable $M_W r$, and the radius of the monopole $R_{\rm m}$ is of order $\sim 1/M_W$. The corresponding energy of the Lagrangian (2.1) is

$$E = \int d^3r \left[\frac{1}{2}B_i^a B_i^a + \frac{1}{2}D_i\phi^a D_i\phi^a + V(\phi)\right] \ , \tag{2.5}$$

where $B_i^a = \frac{1}{2}\epsilon_{ijk}F_{jk}^a$ is the non-abelian magnetic field. The differential equations to determine the profile functions are obtained by minimizing the energy functional and the mass of the monopole is obtained by evaluating the energy at its minimum. The result is

$$M_{\rm m} = \frac{4\pi v}{e} f(\lambda/e^2) \ , \tag{2.6}$$

where f is a slow varying monotonic function that satisfies $f(0) = 1$ and $f(\infty) = 1.787$.

Using the Bogomol'nyi trick [10] we can obtain a lower bound on the mass of the monopole

$$\begin{aligned} M_{\rm m} &\geq \int d^3r \left[\frac{1}{2}B_i^a B_i^a + \frac{1}{2}D_i\phi^a D_i\phi^a\right] \\ &= \int d^3r \frac{1}{2}(B_i^a \pm D_i\phi^a)^2 \mp \int d^3r B_i^a D_i\phi^a \\ &\geq \frac{4\pi v}{e}|n| \ . \end{aligned} \tag{2.7}$$

Where n is the topological degree of the map from $\mathbf{S}^2$ at spatial infinity to the vacuum manifold $\mathbf{S}^2$. The limit where the potential is sent to zero, while keeping fixed the vev v, is the so called BPS limit. In this limit an exact solution for the monopole of charge 1 was first obtained by Prasad and Sommerfield in [11]. This solution saturates the Bogomol'nyi bound.

Now we come to multi-monopoles. It was pointed out in [12] that multi-monopoles with topological charge greater then 1 cannot be spherical symmetric. This has to do with the topology of the maps of $\mathbf{S}^2$ onto itself. Call ϕ a generic map, it is defined spherical symmetric if

$$\phi = g\phi g^{-1} \ , \tag{2.8}$$

for every choice of the isometry $g \in SO(3)$ of the sphere $\mathbf{S}^2$. The only spherical map that exist is in the topological sector $n = 1$ and is the identity. This is the map of the

’t Hooft-Polyakov monopole at infinity (2.4). The maximal symmetry that we can have for multi-monopoles is the axial one. An exact solution for the axial symmetric monopole of charge 2 has been found by Ward in [13]. Thereafter Prasad [15] found a method to construct an axial symmetric monopole of generic charge n.

3 Multi-monopoles are Magnetic Bags

In this section we are going to consider multi-monopoles in the large n limit. As in the case of multi-vortices, a great simplification occurs when the number of magnetic flux is large enough. The soliton becomes a bag, which is a domain wall of negligible thickness that separates an internal region where $\phi = 0$ from an external region where the theory is in the abelian phase. In this limit we can compute the mass and the profile functions using simple arguments.

3.1 The magnetic bag

Our claim is that, for sufficiently large n, multi-monopoles become magnetic bags such as the one shown in Figure 2. In the internal region $\vec{\phi} = 0$. This is an instable

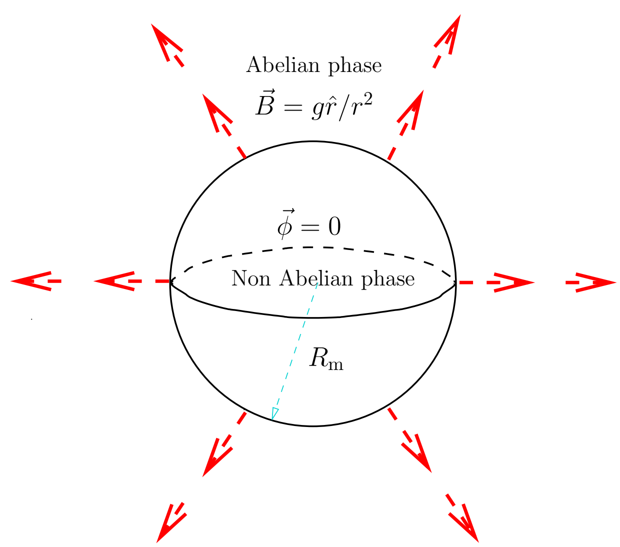

Figure 2: The magnetic bag. The sphere $\mathcal{S}^2$ separates two regions: the internal one is in the non-abelian false vacuum $\phi = 0$, the external one is in the abelian phase.

stationary point of the potential in Figure 1 with energy density $\varepsilon_0 = \lambda v^4/8$. This vacuum is in an abelian Coulomb phase and the magnetic field is given by the curl of the Wu-Yang vector potential [19]:

$$A^N_\varphi = g(1-\cos\theta) \ , \qquad 0 \leq \theta \leq \frac{\pi}{2} + \epsilon \ ,$$
$$A^S_\varphi = -g(1+\cos\theta) \ , \qquad \frac{\pi}{2} - \epsilon \leq \theta \leq \pi \ . \tag{3.1}$$

The matching of the vector bundle gives the Dirac quantization condition:[1]

$$eg = n \tag{3.2}$$

In the external region there is a magnetic field equivalent to the one generated by a magnetic charge g uniformly distributed on the bag surface:

$$\vec{B} = g\frac{\hat{r}}{r^2} \ , \qquad r \geq R \ . \tag{3.3}$$

The energy stored in the magnetic field is:

$$E_B = \int \frac{\vec{B}^2}{2} = \int_R^\infty 4\pi r^2 \, dr \, \frac{g^2}{2r^4} = \frac{2\pi g^2}{R} \ . \tag{3.4}$$

The $1/R$ dependence means that there is a negative pressure outside the monopole that tends to expand the bag. To obtain a stable object we need an opposite force that tends to squeeze the bag. There are three possible sources for this force: the energy stored in the tail of the scalar field, the tension of the wall $T_{\rm W}$ and the internal energy density ε_0.[2] The radius of the magnetic bag $R_{\rm m}$ is determined by the balance between these opposite oriented forces.[3]

3.2 Multiple zeros

We can understand why multi-monopoles become magnetic bags using the analogy with the multi-vortex case. There are two fundamental reasons why the profile of

[1]We have already taken in account the fact that the charge of a fundamental fermion is not e but $e/2$.

[2]This is nothing but the Derrick [17] collapse force coming from the scalar part of the action $\partial\phi\partial\phi + V(\phi)$.

[3]The fact that 't Hooft-Polyakov monopoles become magnetic bags in the large n limit can also be generalized to the Julia-Zee dyon [18].

the multi-vortex becomes a step function. When r is greater than the radius of the vortex, the scalar field ϕ approaches the vev exponentially

$$\phi(r) - v \propto e^{-M_H r} , \qquad r \gg R . \tag{3.5}$$

Near the center it is instead polynomial

$$\phi(r) \propto r^n , \qquad r \ll R \tag{3.6}$$

where n is the number of magnetic fluxes. The only possible way to match together (3.5) and (3.6) is a step function in the large n limit. Now what about the multi-monopole? (3.5) is still true but (3.6) must be discussed more carefully. In the multi-vortex case, (3.6) is a consequence of the fact that the Higgs field winds n times at infinity and at $r = 0$ must vanish and must be analytic.

In the case of the multi-monopole we have to consider the maps from $\mathbf{S}^2$ to $\mathbf{S}^2$ and study their analytic behaviour near zero. The simplest (i.e. the most symmetric) case is the axial symmetric multi-monopole. The map is represented in Figure 3. The

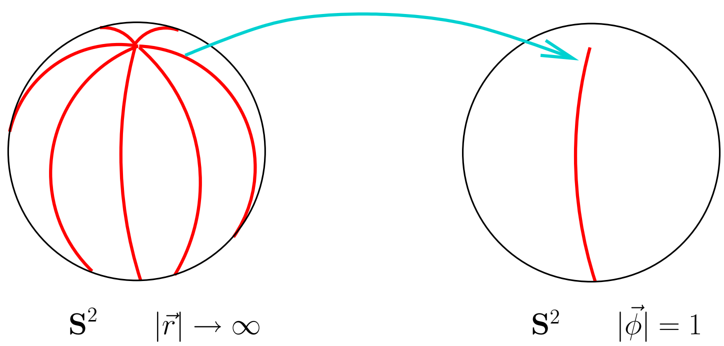

Figure 3: The axial symmetric map from the sphere $\mathbf{S}^2$ at spatial infinity to the sphere $\mathbf{S}^2$ of the vacuum manifold.

norm of the Higgs field is $\phi = \sqrt{\phi_1^2 + \phi_2^2 + \phi_3^2}$ and the three components have the following behaviour near zero:

$$\phi_3 \sim z , \qquad \phi_1 + i\phi_2 \sim (x + iy)^n , \tag{3.7}$$

where n is the winding number. Now we take a particular plane passing through the origin and we want to evaluate the polynomial behavior of the ϕ field near spatial zero. For the axial plane, the plane that is perpendicular to the axial line, the component

ϕ_3 vanishes while the complex vector $\phi_1 + i\phi_2$ winds n times around zero. This means that, on the axial plane, the norm of the Higgs field is $\propto r^n$ near the origin and the extra dimension of the bag can be developed. (The magnetic bag is essentially this "fat" zero.) If we consider any other plane that is not the axial one, we have to consider also the winding of the vectors $\phi_2 + i\phi_3$ and $\phi_3 + i\phi_1$ and then the norm of the Higgs field vanishes linearly. The result is that in the $n \to \infty$ limit we have a magnetic disc that lies in the axial plane. The thickness of the disc is of order $1/M_W$ while its radius is of order n/M_W. We will analyse in more detail the magnetic disc in Section 5 when we check our statements confronting with the exact known solution of the axial symmetric multi-monopole.

For the moment we are interested in a more generic situation where for every choice of the plane that passes through the origin, the norm of the Higgs field vanishes with some power of order n, that is $\phi \propto r^{\mathcal{O}(n)}$. We can also imagine that is possible to find a limit where there is no preferred direction and the various patches that cover the $\mathbf{S}^2$ sphere are distributed in a *homogeneous* way.[4] In this way the multi-monopole should recover the spherical symmetry at $n \to \infty$ and should become a spherical magnetic bag. To see that this is in fact possible we remand to the discussion on the Jarvis rational map in Subsection 5.2.

3.3 The stabilizing forces and the spectrum of multi-monopoles

When the potential vanishes, both the tension of the wall and the internal vacuum energy density are zero. The stabilizing force comes only from the long tail of the Higgs field ϕ outside the monopole. Since the potential is zero, the mass of the scalar field vanishes and then the profile $\phi(r)$ approaches the vev v like $1/r$ and not exponentially. So the scalar field profile is:

$$|\phi| = 0 \qquad r \leq R \; ,$$

$$|\phi| = v\left(1 - \frac{R}{r}\right) \qquad r \geq R \; . \tag{3.8}$$

The energy coming from this tail is:

$$\int \frac{1}{2}\partial_r\phi\partial_r\phi = 2\pi v^2 R \; . \tag{3.9}$$

[4] This is in fact the case we are going to consider in Section 8.1 where we address the possible production of multi-monopoles in the cosmological context.

This brings a pressure force (from outside) that tends to contract the bag and balances with the expansion force coming from the magnetic field (3.4).

Proposition 1 *The magnetic bag, in the limit of vanishing potential, saturates the Bogomol'nyi bound.*

Proof. The mass bag formula is

$$M(R) = \frac{2\pi n^2}{e^2 R} + 2\pi v^2 R \tag{3.10}$$

minimizing with respect to R we obtain

$$R_{\rm BPS} = \frac{n}{ev} \ , \tag{3.11}$$

and inserting back into (3.10)

$$M_{\rm BPS} = \frac{4\pi v}{e} n \tag{3.12}$$

we obtain exactly the BPS mass. ∎

Now we introduce a tiny potential (2.2) and the mass bag formula is

$$M(R) = \frac{2\pi n^2}{e^2 R} + 2\pi v^2 R + T_{\rm W} 4\pi R^2 + \varepsilon_0 \frac{4}{3}\pi R^3 \ , \tag{3.13}$$

where the tension of the wall is $T_{\rm W} \sim \sqrt{\lambda} v^3$, its thickness is $\Delta_{\rm W} \sim 1/(\sqrt{\lambda} v)$ and the internal energy density is $\varepsilon_0 = \lambda v^4/8$. The mass bag formula can have three different regimes. The first regime is the BPS one where the tension scales linearly with n. Then we have a SLAC regime where we neglect the zero energy density ε_0 and finally the MIT regime where we neglect the tension of the wall and consider only the internal energy density. In the case of a quartic potential such as (2.2) we have only two regimes the BPS and the MIT ones.[5] So for our purposes is enough to compute the radius and the mass in the MIT bag regime where we consider only the first and the last terms in (3.13). The minimization gives

$$R_{\rm MIT} = 2^{1/2} \frac{1}{e^{1/2} \lambda^{1/4} v} \ n^{1/2} \ , \qquad M_{\rm MIT} = \frac{2^{5/2}\pi}{3} \frac{\lambda^{1/4} v}{e^{3/2}} \ n^{3/2} \ . \tag{3.14}$$

[5] The previous analysis of the spectrum can be applied also to a generic potential. The transition between the BPS and the MIT regimes happens at $n^* \sim 2e\frac{v^2}{\sqrt{\varepsilon_0}}$. The only changes can happen in the case of a potential where the non-abelian phase $\phi = 0$ is strongly metastable. In this case we should also take in account the presence of a SLAC window between the two regimes.

We are finally ready to analyze the complete spectrum of multi-monopoles. In Figure 4 we show the curves $M_{\rm BPS}(n)$ and $M_{\rm MIT}(n)$ that intersect at

$$n^* = \frac{9}{2}\frac{e}{\sqrt{\lambda}} \ . \tag{3.15}$$

Note that in the transition region the radius of the monopole is $R^* \sim 1/(\sqrt{\lambda}v)$ that is exactly the inverse of the Higgs boson mass $1/M_H$. In the BPS region multi-monopoles

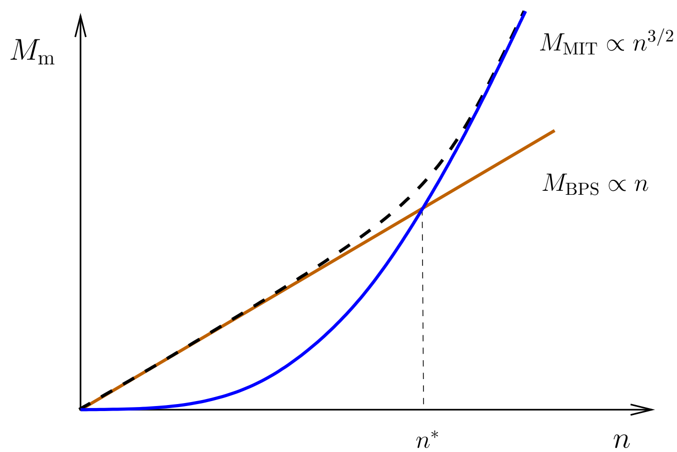

Figure 4: The spectrum of multi-monopoles (black/dashed line) is obtained interpolating between the BPS curve (red/solid line) and the MIT bag curve (blue/solid line). Around $n \sim n^*$ there is a second order phase transition between the BPS regime and the MIT regime.

are marginally stable while in the MIT region they are instable to decaying into monopoles of lower magnetic charge. This imply that multi-monopoles are physical objects only if the potential vanishes or it is very small so that $n^* \gg 1$.

4 Moduli Space of BPS Bags

It is a well known fact that BPS solitons admit a moduli space of solutions. In particular the mass of the n-soliton is equal to the sum of its constituent 1-soliton masses:

$$M_{\rm BPS}(n) = n\, M_{\rm BPS}(1) \ . \tag{4.1}$$

The aim of this section is to describe the moduli space in the large n limit of ANO vortices and 't Hooft-Polyakov monopoles.

4.1 Moduli space of the wall vortex

The moduli space of n BPS vortices, that we indicate by $\mathcal{V}_n$, is a $2n$ real-dimensional space [21, 22]. In the large n limit multi-vortices become bags and the tension bag formula is

$$T(R) = \frac{2\pi n^2}{e^2 R^2} + \varepsilon_0 \pi R^2 \ . \tag{4.2}$$

The BPS potential is $V(|q|) = \frac{e^2}{2}(|\phi|^2 - \xi)^2$ and the Coulomb vacuum energy density is thus $\varepsilon_0 = e^2\xi^2/2$. Minimizing (4.2) with respect to R we obtain exactly the BPS tension $T_{\rm V} = 2\pi n\xi$. The bag formula (4.2) refers to a circle of radius R. If we substitute the circle with a generic surface of area $\mathcal{A}$, we obtain

$$T(R) = \frac{(2\pi n)^2}{2e^2\mathcal{A}} + \varepsilon_0 \mathcal{A} \ . \tag{4.3}$$

Any surface that has the same area of the circular wall vortex, that is $\mathcal{A}_{\rm V} = \pi(R_{\rm V})^2$, has also the same tension. The moduli space of wall vortex is thus the set of surfaces in the plane with fixed area (see Figure 5). Note that, as expected from the large n

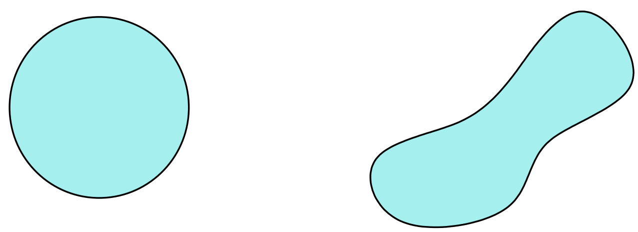

Figure 5: The moduli space for the BPS wall vortex. Two closed surfaces with the same area represent two vortices with the same tension.

limit, this is an infinite dimensional moduli space. It is also easy to see that for the non-BPS vortex the moduli space is lifted by the contribution of the tension of the wall $T_{\rm W} 2\pi R$. When the wall tension is added, the minimal tension for the vortex is obtained when the perimeter is minimized keeping the area fixed. This implies that the wall vortex of minimal tension is the circular one.

4.2 Moduli space of the BPS magnetic bag

Now we return to the main subject of the paper: multi-monopoles and magnetic bags. The moduli space of n BPS monopoles, that we denote with $\mathcal{M}_n$, is a $4n$

real-dimensional manifold (we are referring to the $SU(2)$ case) [21, 22, 64]. The $4n$ coordinates can be interpreted as the positions of 1-monopoles plus the $U(1)$ phase factor.

Now we derive the moduli space of the BPS magnetic bag. We will find that there are an infinite set of closed surfaces $\mathcal{S} \subset \mathbf{R}^3$ with the same mass of the spherical surface. It is convenient to introduce the magnetic scalar potential such that $\vec{B} = \vec{\nabla}\varphi$. The Maxwell equation $\vec{\nabla} \cdot \vec{B} = 0$ is thus transformed into the Laplace equation $\Delta\varphi = 0$, while the other Maxwell equation $\vec{\nabla} \wedge \vec{B} = 0$ is automatically satisfied. The magnetic potential outside the bag $\mathcal{S}$ satisfies the Laplace equation with the following boundary conditions:

$$\varphi|_{\mathcal{S}} = \text{const}, \qquad \int \left.\frac{\partial\varphi}{\partial n}\right|_{\mathcal{S}} = 4\pi g \ , \qquad \varphi|_{\infty} = 0 \ . \tag{4.4}$$

The scalar field ϕ satisfies also the Laplace equation but with different boundary conditions:

$$\phi|_{\mathcal{S}} = 0 \ , \qquad \phi|_{\infty} = v \ . \tag{4.5}$$

The mass of the bag is the sum of the energy of the magnetic field plus the energy of the scalar field:

$$M(\mathcal{S}) = \int \frac{(\vec{\nabla}\varphi)^2}{2} + \int \frac{(\vec{\nabla}\phi)^2}{2} \tag{4.6}$$

Let us recall for a moment the case of a spherical bag. If we denote R the radius of the sphere, the potential φ and the scalar field ϕ are given by

$$\varphi = -\frac{g}{r} \ , \qquad \phi = v\left(1 - \frac{R}{r}\right) \ . \tag{4.7}$$

The mass as function of the radius is

$$M(R) = \frac{2\pi g^2}{R} + 2\pi v^2 R \ , \tag{4.8}$$

and the minimization gives

$$R_{\rm m} = \frac{g}{v} \ , \qquad m_{\rm m} = 4\pi g v \ . \tag{4.9}$$

Proposition 2 *There is an infinite set of bags with different shapes that saturates the Bogomol'nyi bound.*

Proof. To construct the generic magnetic bag we need the following trick. Consider a generic function from the ball $\mathbf{B}^3$ of radius 1 to the space $\mathbf{R}^3$:

$$f : y \in \mathbf{B}^3 \longrightarrow \mathbf{R}^3 \tag{4.10}$$

and then solve the Laplace equation with a source given by the image of the function $\vec{f}$:

$$\Delta\varphi(x) = 4\pi g \int_{\mathcal{B}^3} \delta^{(3)}\left(x - f(y)\right) \ \det\left(\frac{\partial f}{\partial y}\right) \ d^3y \tag{4.11}$$

Physically we can think of it in this way. We have a magnetic source of charge g that is distributed on a compact region $f(\mathcal{B}^3)$ with generic shape and generic distribution of magnetic charge. In Figure 6 (a) we have the blob and the surfaces of constant ϕ. In the following we will see that a particular surface of this set give rise to a magnetic bag with the same mass as that of the spherical bag. Consider one of the surfaces of

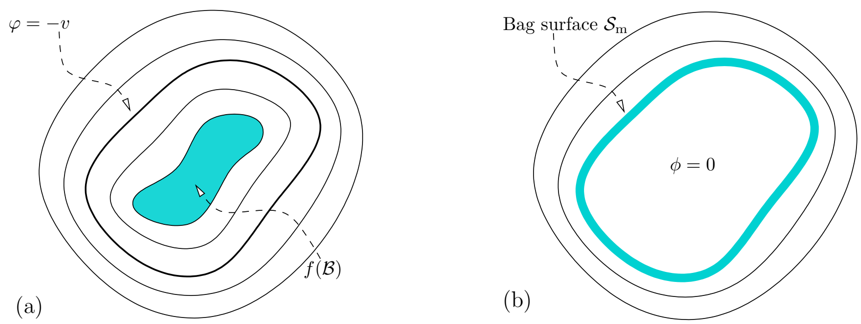

Figure 6: (a): $f(\mathbf{B}^3)$ is a compact source of magnetic field. The closed surface are the ones where the magnetic scalar potential φ is constant. For $f(\mathbf{B}^3)$ sufficiently compact, there will exist a particular surface where the value of the potential φ is equal to $-v$. (b): The magnetic bag is obtained taking the surface given by the previous construction and then distributing on it a magnetic charge equivalent to the one in the blob of (a). (We stress that (a) is the mathematical construction while (b) is the physical result.)

the Figure 6 (a). We want a magnetic bag where φ is given by the solution of (4.11) outside this surface. Note that this automatically satisfies the boundary conditions (4.4). If we want to minimize the energy we have to satisfy the abelian BPS equations

$$\vec{\nabla}\varphi = \vec{\nabla}\phi \ . \tag{4.12}$$

To do so we simply have to choose the particular surface where

$$\varphi|_{\mathcal{S}_{\rm m}} = -v \ . \tag{4.13}$$

This is the central point of the construction. It is easy to see that the following choice for the Higgs field

$$\phi = v + \varphi \tag{4.14}$$

satisfies the Laplace equation and the boundary conditions (4.5) and also the abelian BPS equation (4.12).

A check that the energy saturates the Bogomol'nyi bound is given by a simple application of the Green's first identity [67]

$$\int_V \frac{(\vec{\nabla}\varphi)^2}{2} = -\frac{1}{2}\int_{\mathcal{S}} \varphi \frac{\partial \varphi}{\partial \vec{n}} = -2\pi g \left.\varphi\right|_{\mathcal{S}} \ . \tag{4.15}$$

and so

$$\int_V \frac{(\vec{\nabla}\varphi)^2}{2} = \int_V \frac{(\vec{\nabla}\phi)^2}{2} = 2\pi g v \ . \tag{4.16}$$

The sum of the two is $4\pi g v$ which is exactly the Bogomol'nyi bound. ■

The knowledge of the moduli spaces $\mathcal{V}_n$ and $\mathcal{M}_n$ is essential to describe the low energy dynamics of vortices and monopoles. The motion of these particles is described by geodesics in the moduli space [23]. It is thus fundamental to know not only the topology but also the metric of these spaces. For the monopoles the situation is better since we know that $\mathcal{M}_n$ is an hyper-Kähler manifold. This enabled Atiyah and Hitchin to find the exact metric of the moduli space of two monopoles and consequently to describe their scattering [24]. The same process for vortices can be described only using numerical computations [26, 27]. A method of Manton permits one to obtain by simple arguments the metric of monopoles and vortices when they are far apart [25]. It is interesting to note that our result shed light on a complete opposite regime, when a large number of solitons are very close to each others.

We want to make another conclusive remark to this section. The large n limit we are studying in this paper can be seen as a sort of linearization of the problem. The "non-abelianity" is restricted only to a small shell of thickness negligible compared to the radius of the bag. The essential parameter, such as the radius R_{m} and the mass M_{m} of the multi-monopoles, can be computed only using the abelian theory outside the bag. This suggest an intriguing relation with the linearization of the Nahm equations in the $n \to \infty$ limit found in [16].

5 More on the Magnetic Bag

This section has a double aim. First we want to clarify better the magnetic bag conjecture in the case of the BPS monopole. Second we want to test the conjecture using the Ward-Prasad-Rossi (WPR) [13][15] solution for the axial symmetric multi-monopole.

For simplicity we work in units where $e = v = 1$ and the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2}D_\mu\phi^a D^\mu\phi^a \ . \tag{5.1}$$

The non-abelian BPS equations are

$$B^a_i = \pm D_i\phi^a \tag{5.2}$$

where $B^a_i = \frac{1}{2}\epsilon_{ijk}F^a_{jk}$ is the non-abelian magnetic field. The 't Hooft-Polyakov monopole has the following structure

$$\phi^a = \hat{r}^a H(r) \ , \qquad A^a_i = -\epsilon_{aij}\frac{\hat{r}^j}{r}G(r) \tag{5.3}$$

and the profile functions are

$$H(r) = \coth r - \frac{1}{r} \ , \qquad G(r) = 1 - \frac{r}{\sinh r} \ . \tag{5.4}$$

Using the identity $\partial_i\hat{r}_j = (\delta_{ij} - \hat{r}_i\hat{r}_j)/r$ we can compute the non-abelian magnetic field

$$B^a_i = \underbrace{-\hat{r}^a\hat{r}^i H'(r)}_{\text{Abelian}} + \underbrace{\left(\delta_{ia} - \hat{r}^a\hat{r}^i\right)\frac{H(r)}{\sinh r}}_{\text{Non}-\text{Abelian}} \ . \tag{5.5}$$

The first piece is the magnetic field projected in the $\phi(r)$ direction (note that it vanishes like $1/r^2$ at infinity) and thus corresponds to the abelian magnetic field of the unbroken $U(1)$. The second term is a pure non-abelian field and vanishes exponentially at infinity.

Now suppose that we have all the multi-monopole solutions that we want, how we can check the magnetic bag conjecture? The idea is simple and it is the same we have used for the multi-vortex in [3]. While we are making the large n limit we also rescale the lengths such that the radius of the monopole remains constant $r \to r/n$. In this case, if the magnetic bag conjecture is true, it would appear very clearly as in Figure 7 for the norm of the Higgs field and Figure 8 for the abelian part of the magnetic field.

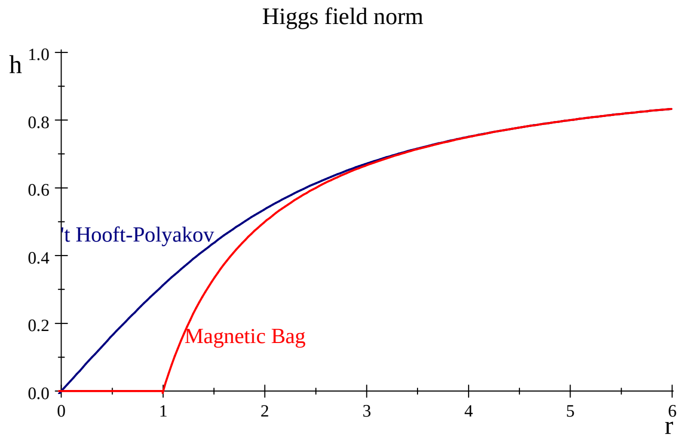

Figure 7: The norm of the Higgs field for the $n = 1$ monopole and for the $n = \infty$ monopole. The radius of the multi-monopole is kept fixed while making the large n limit.

Unfortunately not all the multi-monopole solutions are known. But one solution is known in great detail: the axial symmetric multi-monopole. In what follows we are going to confront our conjecture with this solution and we will find an encouraging agreement with the expectations.

5.1 The axial symmetric multi-monopole

The WPR axial symmetric multi-monopole is an exact solution for monopoles of multiple magnetic charge. The axial symmetry can be seen in Figure 3 where we have the map from the sphere $\mathbf{S}^2$ at spatial infinity to the sphere $\mathbf{S}^2$ of the vacuum manifold $|\vec{\phi}| = 1$. The spatial sphere covers the vacuum manifold sphere n times winding around the $\hat{z}$ axis.

The $n = 1$ case corresponds to the Prasad-Sommerfield [11] solution while the $n = 2$ case is the multi-monopole of charge 2 first obtained in [13]. The generalization to arbitrary n has been found in [15]. The axial symmetric multi-monopole corresponds to one particular point in the moduli space of multi-monopoles. Say in another way, the imposition of the axial symmetry fixes all the degrees of freedom in the moduli space, apart from a global translation and rotation.

Now let's find the exact value of the radius of the magnetic disc. To find it we must repeat the procedure we have done for the magnetic sphere. We take a generic

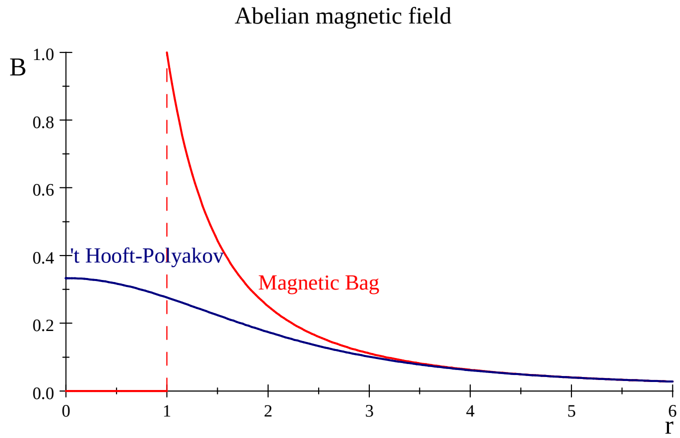

Figure 8: The abelian projection of the magnetic field for the $n = 1$ monopoles and for the $n = \infty$ monopole. The radius of the multi-monopole is kept fixed while making the large n limit.

magnetic disc of radius R_{d} with magnetic charge 1. The Higgs field ϕ is a solution to the Laplace equation with Dirichlet boundary conditions: $\phi = 0$ on the disc and $\phi = 1$ at infinity. The solution can be written in cylindrical coordinates using the expansion in Legendre polynomials [67]:

$$\phi(r,\theta) = \begin{cases} 1 - \frac{2}{\pi}\frac{R_{\mathrm{d}}}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R_{\mathrm{d}}}{r}\right)^{2l} P_{2l}\left(\cos\theta\right) \ , & r \geq R_{\mathrm{d}} \\ \\ \frac{2}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{r}{R_{\mathrm{d}}}\right)^{2l+1} P_{2l+1}\left(\cos\theta\right) \ , & r \leq R_{\mathrm{d}} \end{cases} \tag{5.6}$$

The magnetic scalar potential φ is again a solution to the Laplace equation but with different boundary conditions: $\varphi = 0$ at infinity and $\varphi = \mathrm{const}$ on the disc, where the constant is fixed imposing that the charge of the disc is 1. The solution is

$$\varphi\left(r,\theta\right) = \begin{cases} \frac{1}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R_{\mathrm{d}}}{r}\right)^{2l} P_{2l}\left(\cos\theta\right), & r \geq R_{\mathrm{d}} \\ \\ \frac{\pi}{2R_{\mathrm{d}}} - \frac{1}{R_{\mathrm{d}}}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{r}{R_{\mathrm{d}}}\right)^{2l+1} P_{2l+1}\left(\cos\theta\right), & r \leq R_{\mathrm{d}} \end{cases} \tag{5.7}$$

where it turns out that $\varphi = \frac{\pi}{2R_{\mathrm{d}}}$ on the disc. To obtain the radius of the disc we must sum the energy carried by the two scalar potentials and then minimize with respect to R_{d} (the same we have done in (3.10) for the spherical bag). But we can use a shortcut since we know that the energy is minimized exactly when the two contribution are equal (see the abelian BPS equation (4.12)), and we easily obtain the radius of the

disc:

$$R_{\rm d} = \frac{\pi}{2} \ . \tag{5.8}$$

The magnetic charge on the disc has distribution $\frac{1}{\pi^2}\frac{1}{\sqrt{\left(\frac{\pi}{2}\right)^2-\left(x_1^2+x_2^2\right)}}$. The magnetic disc (Figure 9) is consistent with what we discussed in Section 4. The shape of the bag is now degenerate and squeezed in the x_3 direction.

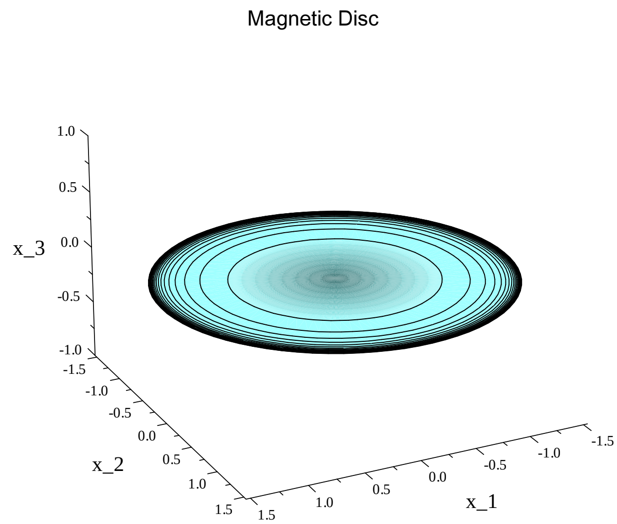

Figure 9: The magnetic disc centered in zero, with radius $\frac{\pi}{2}$ and with the axial line oriented in the x_3 direction. The density of the concentric lines is proportional to the density of magnetic charge.

Now it is time to confront with the WPR solution. We will not give the details of the derivation, the reader can find them in the literature (in particular in Ref. [15]). We will just present the formulas that are needed for the solution. First of all we have to introduce the functions $\Delta_{n,l}(x_{i=1\dots4})$

$$\Delta_{n,l}(x_{i=1\dots4}) = \frac{1}{2}(-1)^l e^{ix_4 - il\theta} \int_{-1}^{1} \left[2\cos\left(\frac{\pi t}{2}\right)\right]^{n-1} e^{-x_3 t} \left(\frac{1+t}{1-t}\right)^{l/2} I_l\left(s\sqrt{1-t^2}\right) dt \tag{5.9}$$

where n is the winding number, l is an integer that goes from $-n$ to n, $x_{1,2,3}$ are the spatial coordinates, s is the complex variable $x_1 + ix_2$ and x_4 is the Euclidean time. We can thus introduce the three potentials ϕ_n,ρ_n and χ_n that are needed for

the construction:

$$\phi_n\left(x_{i=1\dots4}\right) = (-1)^{n+1} \frac{\det \begin{pmatrix} \Delta_{n,-n+1} & \cdots & \cdots & \Delta_{n,0} \\ \vdots & & & \vdots \\ \\ \Delta_{n,0} & \cdots & \cdots & \Delta_{n,n-1} \end{pmatrix}}{\det \begin{pmatrix} \Delta_{n,-n+2} & \cdots & \Delta_{n,0} \\ \vdots & & \vdots \\ \Delta_{n,0} & \cdots & \Delta_{n,n-2} \end{pmatrix}}, \tag{5.10}$$

$$\rho_n\left(x_{i=1\dots4}\right) = (-1) \frac{\det \begin{pmatrix} \Delta_{n,-n} & \cdots & \cdots & \Delta_{n,-1} \\ \vdots & & & \vdots \\ \\ \Delta_{n,-1} & \cdots & \cdots & \Delta_{n,n-2} \end{pmatrix}}{\det \begin{pmatrix} \Delta_{n,-n+2} & \cdots & \Delta_{n,0} \\ \vdots & & \vdots \\ \Delta_{n,0} & \cdots & \Delta_{n,n-2} \end{pmatrix}}, \tag{5.11}$$

$$\chi_n\left(x_{i=1\dots4}\right) = (+1) \frac{\det \begin{pmatrix} \Delta_{n,-n+2} & \cdots & \cdots & \Delta_{n,1} \\ \vdots & & & \vdots \\ \\ \Delta_{n,1} & \cdots & \cdots & \Delta_{n,n} \end{pmatrix}}{\det \begin{pmatrix} \Delta_{n,-n+2} & \cdots & \Delta_{n,0} \\ \vdots & & \vdots \\ \Delta_{n,0} & \cdots & \Delta_{n,n-2} \end{pmatrix}}. \tag{5.12}$$

In terms of these potentials it is possible to write the solution of the non-abelian BPS equations (that is the potentials A_i and ϕ). The problem is that this solution is expressed in a complex gauge and, even if it is proven to be gauge equivalent to a real one, the explicit form for the real solution is not known. Fortunately there exist a simple expression for the modulus of the Higgs field

$$h_n\left(r, x_3\right) = \left| \frac{1}{\phi_n} \sqrt{\left(\frac{\partial \phi_n}{\partial x_3}\right)^2 - \left(\rho_n - \frac{\partial \rho_n}{\partial x_3}\right)\left(\chi_n + \frac{\partial \chi_n}{\partial x_3}\right)} \right|. \tag{5.13}$$

This expression gives the modulus of the Higgs field as function of the radius $r = \sqrt{x_1^2 + x_2^2}$ and the coordinate x_3. There are two convenient simplifications of this

formula. On the axial plane (5.13) reduces to

$$h_n(r,0) = \left| \frac{1}{\phi_n} \left(\rho_n - \frac{\partial \rho_n}{\partial x_3} \right) \right|_{x_3=0} = \left| \frac{1}{\phi_n} \left(\chi_n + \frac{\partial \chi_n}{\partial x_3} \right) \right|_{x_3=0} . \tag{5.14}$$

On the axial line (5.13) reduces to

$$h_n(0,x_3) = \left| \partial_{x_3} \ln \Delta_{n,l}(0,0,x_3,0) \right| . \tag{5.15}$$

Now we are ready for the final step: the computation. Using a numerical computation[6] we have been able to find the norm of the Higgs field up to $n=3$. The hardest step for the computation is the derivative with respect to x_3, and for this reason we are able only to go to $n=3$. Anyway the results give a quite convincing proof of the conjecture. In Figure 10 we present the norm of the Higgs field on the plane perpendicular to the axial line using the formula (5.14) while in Figure 11 we present the norm on the axial line using the formula (5.15). The points in the plots correspond to the norm for $n=1,2,3$ (respectively green/dot, blue/circle and red/cross). The gray line is instead the norm of the Higgs field for the magnetic disc computed using the formula (5.6) with $R_{\mathrm{d}} = \frac{\pi}{2}$. On the axial line $r=0$ the convergence can be expressed in a compact mathematical form

$$\lim_{n\to\infty} \frac{\int_{-1}^{1} z e^{-ntx_3} \cos^{n-1} \frac{\pi t}{2} dt}{\int_{-1}^{1} e^{-ntx_3} \cos^{n-1} \frac{\pi t}{2} dt} = \frac{2}{\pi} \arctan \frac{2x_3}{\pi} , \tag{5.16}$$

where the right hand side simply is (5.15) and the left hand side is the resummation of the series (5.6). Formula (5.16) can be verified with great accuracy but we don't know a simple analytic proof of it.[7]

5.2 Large n limit of multi-monopoles

To understand completely the large n limit of multi-monopoles, we have to understand in which way the position in the moduli space is related to the shape of the magnetic bag. In this paper we have provided two examples. In the first one we choose a limit in which the spherical symmetry is asymptotically restored in the large n limit and the outcome is a bag with spherical shape and radius n (in units $e = v = 1$). In the

[6] We have used the program Scientific WorkPlace.

[7] In the Figure 11 we have showed only the values $n = 1,2,3$ but formula (5.16) can be verified much beyond. On the axial plane (Figure 10) $n = 3$ is instead our computational limit.

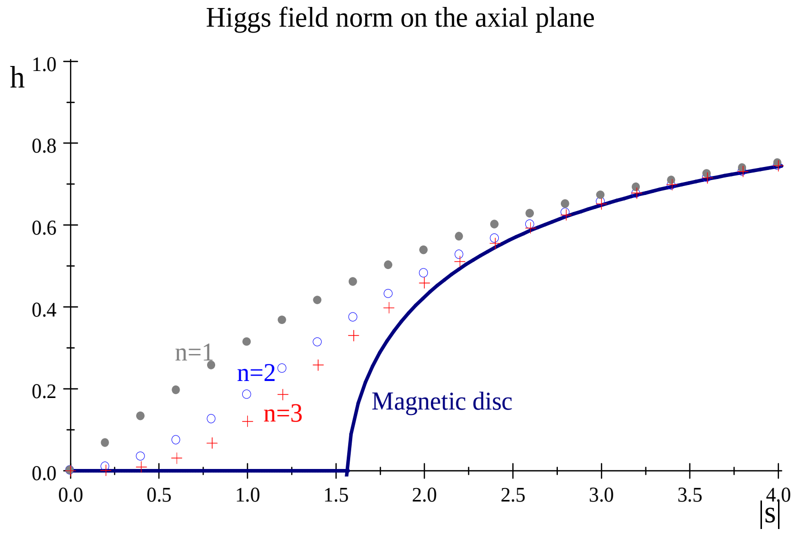

Figure 10: We have plotted the norm of the Higgs field on the axial plane as function of the radius. The dots, circles and crosses refers respectively to $n = 1, 2, 3$, and have been computed using (5.14) with the radius rescaled by a factor of n. The line is the magnetic disc potential and has been computed from (5.6).

second example we have the axial symmetric multi-monopole. The outcome is a bag with a degenerate shape: a disc with radius $\frac{n\pi}{2}$ perpendicular to the axial line of the multi-monopole.

Among the various mathematical approaches to the study of multi-monopoles, the Jarvis [29] rational map is probably the most suitable to the large n limit problem.[8] In the Jarvis framework we have to choose an origin in the space $\mathbf{R}^3$ and then consider the Hitchin equation $(D_r - i\Phi)s = 0$ along each radial line from the origin to infinity. s is a doublet in the $SU(2)$ representation and there is only one solution which decays asymptotically as $r \to \infty$. Call this solution $\begin{pmatrix} s_1(r) \\ s_2(r) \end{pmatrix}$ and $\begin{pmatrix} s_1(0) \\ s_2(0) \end{pmatrix}$ its value at the origin. The Jarvis rational map is given by $R = \frac{s_1(0)}{s_2(0)}$ and is a correspondence between Riemann spheres $R : \mathbf{CP}^1 \mapsto \mathbf{CP}^1$. The map is holomorphic since, due to the Bogomol'nyi equation, the operator $D_r - i\Phi$ commutes with $D_{\bar{z}}$. A gauge

[8]The Jarvis rational map is a modified version of the Donaldson rational map [28]. In the Donaldson we have to choose a particular direction in $\mathbf{R}^3$ and this break the rotational invariance. In the Jarvis case we have to choose an origin. This preserves the rotational invariance around the origin.

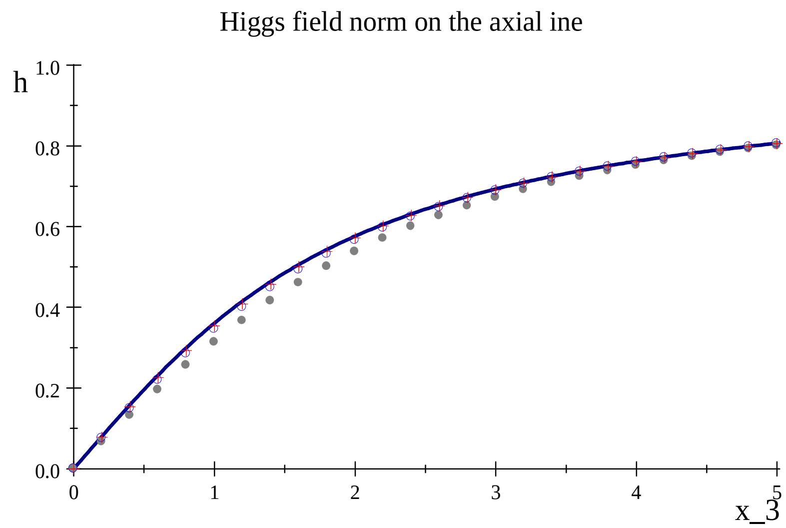

Figure 11: We have plotted the norm of the Higgs field on the axial line as function of the x_3 coordinate. The conventions are the same of Figure 10.

transformation replace R by a $SU(2)$ Mobius transformation determined by the gauge transformation at the origin. We thus have a correspondence between the moduli space of the n-monopole and the equivalence class of unbased rational maps of degree n.[9]

A procedure to obtain the monopole solution from the Jarvis rational map has been studied in [30] and applied to the study of particularly symmetric multi-monopoles. Choosing a certain complex gauge, the boundary conditions for the field ϕ and A_i can be expressed in terms of the rational map. In particular the map from the sphere $\mathbf{S}^2$ at spatial infinity to the sphere $\mathbf{S}^2$ of the vacuum manifold is essentially given by the rational map itself.

Using the Jarvis rational map we can at least speculate what could be a large n limit for multi-monopoles. The map can be thought of as a certain distribution of n zeros and n poles over the Riemann sphere. When n is large the zeros and poles form a continuous distribution. Call $\sigma_0(z)$ and $\sigma_\infty(z)$ the density of zeros and poles over the sphere divided by the total number n. We can define a large n limit so that the densities $\sigma_0(z)$ and $\sigma_\infty(z)$ remain constant. For a homogeneous distribution

[9]The rational map has $4n+2$ parameters. Subtracting the 3 parameters of the $SU(2)$ gauge transformation gives $4n-1$. This is the dimension of gauge inequivalent n-monopoles. In general a $U(1)$ phase is then added for convenience.

$\sigma_0(z) = \sigma_\infty(z) = \frac{1}{4\pi}$ and we should obtain the spherical magnetic bag. The rational map for the axial symmetric multi-monopole is $R(z) = z^n$ and is a highly degenerate distribution with n zeros in zeros and n poles at infinity. It is an open question what is the magnetic bag surface corresponding to generic distributions $\sigma_0(z)$ and $\sigma_\infty(z)$.

6 String Theory Interpretation of Bag Solitons

In this section we will to interpret our result in the string theory context. For the following results we refer in particular to the reviews [61] for brane setups of gauge theories and [60] for solitons.

Monopoles in string theory can be obtained in the following way. We take a stack of N D3-branes in type IIB string theory. The low energy theory that describes the dynamics of the branes is a $\mathcal{N} = 4$ $U(N)$ gauge theory in $3+1$ dimensions. In Figure 12 we have a $U(2)$ gauge theory broken down to $U(1) \times U(1)$. Monopoles correspond to D1-branes stretched between the D3-branes; the point where they end on the brane is the position of the monopole. The brane setup of Figure 12 is just a classical cartoon.

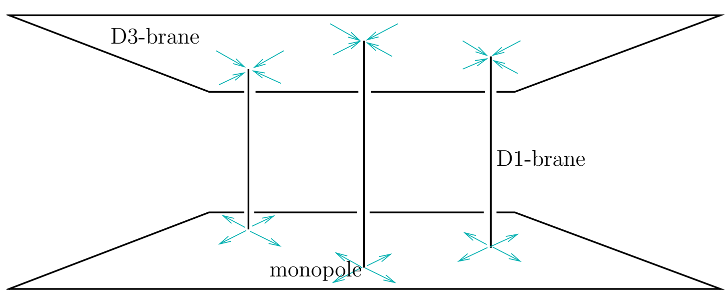

Figure 12: The $\mathcal{N} = 4$ $U(2)$ gauge theory is realized on the world volume of two D3-branes in type IIB string theory. D1-branes stretched between the two D3-branes correspond to BPS monopoles in the four dimensional gauge theory.

When D1-branes end on a D3-brane they create a disturbance in its shape due to their tension. Figure 12 is a reliable approximation only when the distance between the D3-branes is sufficiently large and the monopoles are far enough away from each other so that the disturbances do not overlap. A way to study the disturbance created

by a D1-brane ending on a D3-brane has been developed in [33]. The effective theory that describes the low energy degrees of freedom on a single D3-brane is the Dirac-Born-Infeld (DBI) theory. This non-linear theory possesses non-trivial solutions to the classical equation of motion that can be identified with the D1-brane ending on the D3-brane. This solution, also called a BIon, is a spike coming out of the D3-brane with a profile proportional to $1/r$. This shows that the D1-brane is made of the same substance as that of the D3-brane. When considering D1-branes suspended between two D3-branes we need to use the non-abelian generalization of the DBI action. It has been shown in [34] that for BPS quantities the solution of the non-abelian DBI action is also a solution to its first order expansion (5.1).[10] The suspended D1-brane is thus described by two spikes of D3-branes with profile proportional to $\coth r - \frac{1}{r}$ that meet in a single point. Figure 13 is what happens to the D1-branes when their distance is large enough for the spikes not to be overlapped. In the magnetic bag

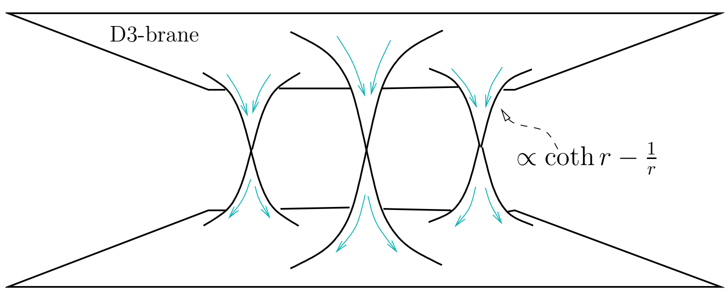

Figure 13: D1-branes suspended between two D3-branes as seen from the DBI action point of view.

limit we are in the opposite regime where the disturbances created by the D1-branes are completely overlapped so it no more is meaningful to speak about the position of the single D1-brane. What emerges is instead the magnetic bag surface $\mathcal{S}_{\rm m}$ of radius $\propto n$ where the two spikes coming out from the D3-branes are joined together (see Figure 2). The two spikes have profiles proportional to $1 - \frac{n}{r}$. In the interior of the magnetic bag the two D3-branes are on the top of each other.

Now we consider wall vortices in the string theory context. Even in this case a

[10]Still there is an uncertainty about the way to take the traces in the non-abelian DBI action. Ref. [34] uses the prescription given in Ref. [35].

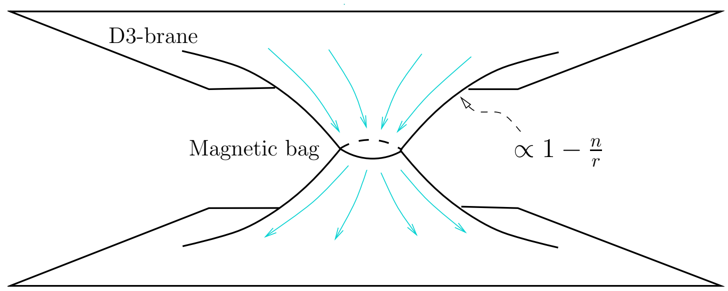

Figure 14: The magnetic bag is a tube of D3-branes connecting the two gauge 3-branes. The magnetic flux passes trough the tube and stabilizes it.

"brane transmutation" effect will be the string theoretical explanation. We will see that by T-duality the wall vortex is essentially the same object as the magnetic bag.

The four dimensional theory is $\mathcal{N} = 2$ SYM with gauge group $U(N_c)$ and N_f matter hypermultiplets with masses m_i. This theory is broken to $\mathcal{N} = 1$ by a superpotential $\mathcal{W}(\Phi)$ for the adjoint chiral superfield. The moduli space of the $\mathcal{N} = 2$ theory is lifted and only a discrete number of vacua survive [37]. We are interested in vacua where some diagonal element of the adjoint scalar field ϕ_j is equal to some flavor mass m_i. Vortices arise in the color-flavor locked vacua.

The brane realization is obtained in type IIA string theory as follows. The $\mathcal{N} = 2$ theory is obtained with two NS5-branes extended in $x^{0,1,2,3,4,5}$ at the positions $x^6 = 0$ and $x^6 = L$ (we call them NS5 and NS5$'$). Then there is a stack of N_c D4-branes extended in $x^{0,1,2,3,6}$ between the two NS5-branes. Finally there is a set of N_f semi-infinite D4-branes that end on the NS5$'$-brane [38]. The breaking to $\mathcal{N} = 1$ is obtained by giving a shape to the NS5$'$ in the $x^{7,8}$ plane, by a quantity proportional to the derivative of the superpotential: $x^6 + ix^7 \propto \mathcal{W}'(x^4 + ix^5)$. The resulting configuration is that of Figure 15. Vortices correspond to D2-branes stretched between a color-flavor locked D4-brane and the NS5$'$-brane.

When a lot of vortices are close to each other we obtain the wall vortex. We expect that even in this case there is a simple classical description in terms of D-branes. Our proposal is the following. The D2-branes expand out and get transformed into a D4-brane like in Figure 16. This D4-brane is extended in the time direction, the three physical dimensions of the wall vortex, and the segment between the locked D4-brane

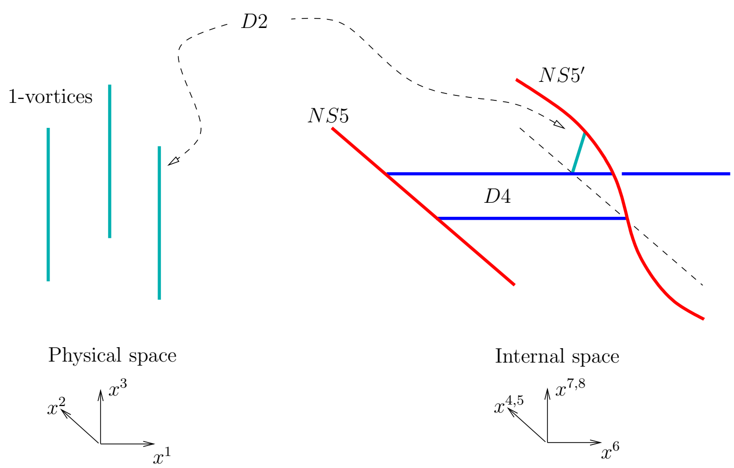

Figure 15: Brane setup for the $\mathcal{N}=2$ theory broken down to $\mathcal{N}=1$ by a superpotential. Vortices in the physical space correspond to D2-branes stretched between the color-flavor locked D4-brane and the NS5′-brane.

and the NS5′-brane. Inside the wall vortex the locked D4-brane is connected to the NS5′-brane and so the $U(1)$ gauge is restored. This explains the presence of the magnetic flux in core of the wall vortex.

Figure 16 is not the end of the story. Now that the D4-brane is reconnected to the NS5′-brane, it tries to minimize its energy. Two cases must be distinguished. If the $\mathcal{N}=1$ breaking is obtained by a superpotential, the D4-brane splits in two pieces, one reaches the nearest root of $\mathcal{W}'(x^4+ix^5)$, and the other remains attached to the NS5′-brane (see first part of Figure 17). If the $\mathcal{N}=1$ breaking is due to a Fayet-Ilipoulos term (or equivalently to a linear superpotential), the D4-brane is lifted as in the second part of Figure 17.

We conclude the section by making a comparison between the string interpretations of the two bag solitons.

- When a lot of D1-branes (1-monopoles) a near to each other they get transformed into a D3-brane (magnetic bag) with a magnetic flux turned on. The two lacking dimensions are given by the bag surface $\mathcal{S}_{\mathrm{m}}$.
- When a lot of D2-branes (1-vortices) are close to each other they become a D4-brane (wall vortex) with a magnetic flux turned on. The two lacking dimensions correspond to the area of the wall vortex $\mathcal{A}_{\mathrm{V}}$.

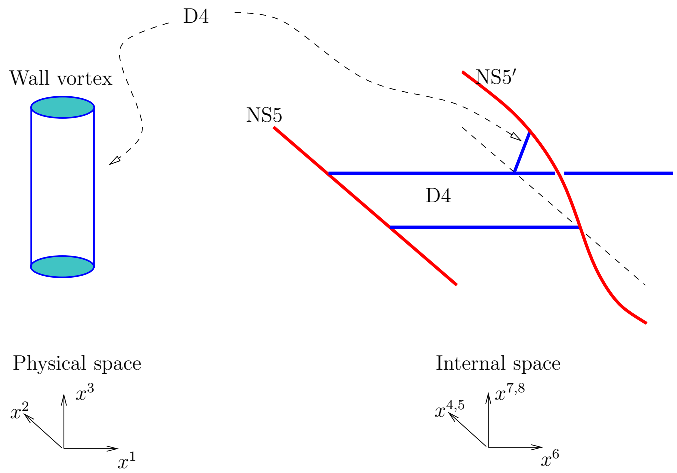

Figure 16: A lot of coincident D2-branes expand out in a D4-brane. In the physical space this corresponds to the wall vortex.

These two phenomena are very similar and in fact, if we lift both configurations to M-theory (x^{10} is compactified on a circle), we discover that they are identical. A lot of M2-branes near to each other get transformed into a M5-brane wrapped on the M-theory circle. On the M5-brane there is a flux $F_{10,i,j}$ turned on where F is the field strength of the bi-form A that lives on the M5-brane. The field $F_{10,i,j}$ is nothing but the magnetic flux that passes through the bag surface $\mathcal{S}_{\mathrm{m}}$, in the case of the magnetic bag, and $\mathcal{A}_{\mathrm{V}}$ in the case of the wall vortex.

7 Multi-monopoles and Gauge Theories

In this section we want to explore the relation between the 't Hooft large n limit of certain $SU(n)$ gauge theories and the large n limit of multi-monopoles discussed in this paper.

Let us start with the simplest case. It has been discovered in [42, 43] that the Coulomb branch of $\mathcal{N} = 4$ $SU(n)$ Yang-Mills theory in $d = 2 + 1$ dimensions is isomorphic, as a hyper-Kahler manifold, to the moduli space of n uncentered BPS monopoles of a $SU(2)$ gauge theory in $d = 3 + 1$ dimensions. A simple explanation of this has been found by Hanany-Witten [44] considering a brane configuration of n horizontal D3-branes suspended between two vertical NS5-branes. Looking at the

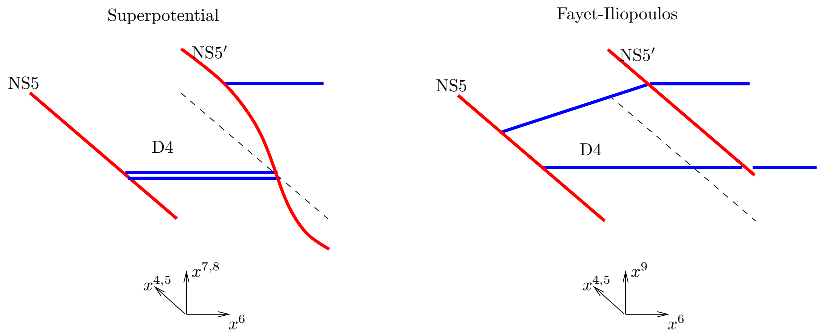

Figure 17: The locked D4-brane of Figure 16 wants to minimize its energy. Two cases must be distinguished: the breaking by a superpotential when there is at least one root of W' and the breaking by a Fayet-Iliopoulos term.

low energy limit of this configuration from a "horizontal" perspective we recover the $SU(n)$ three dimensional gauge theory while, from a "vertical" perspective, we recover the $SU(2)$ gauge theory with n monopoles. The moduli space under consideration has real dimension $4(n-1)$, i.e. four times the rank of the $SU(n)$ gauge group.

Now we want to make the large n limit of the three dimensional gauge theory. According to the 't Hooft prescription, while sending n to infinity we have also to rescale the gauge coupling $\frac{1}{g_{YM}^2} \sim n$ or, equivalently, keep the dynamical scale Λ fixed. But in a theory such as the one under consideration, we have also to specify the sequence of points in the moduli space of vacua. The main problem is that the moduli space changes as n change and so it is not obvious what sequence must be chosen in order to obtain a definite limit. In the theory under consideration the moduli space is that of n BPS monopoles so our guess is that we have to use the prescription adopted in this paper: sending n to infinity while keeping fixed the shape of the magnetic bag. Of course in order to properly define this limit we have to understand the large n limit of multi-monopoles and what is discussed in Subsection 5.2 are just some speculations along this direction.

But one thing we at least can check: the 't Hooft rescaling correspond exactly to the rescaling used in Section 5 where the radius of the magnetic bag is kept fixed while making the large n limit. We have to consider the parameters that enter in the Type IIB brane configuration. We have the string coupling constant g_s, the Regge slope α' which we set to one for convenience, the distance between the two NS5-branes v and the number of D3-branes n. The three dimensional $SU(n)$ gauge theory

knows only about two combination parameters, the size of the gauge group n and the coupling constant $\frac{1}{g_{YM}^2} = \frac{v}{g_s}$. The four dimensional $SU(2)$ gauge theory knows about three parameters: the coupling constant $\frac{1}{e^2} = \frac{1}{g_s}$, the vev of the Higgs field v and the number of monopoles n. If we want to perform the 't Hooft large n limit we have to send $\frac{1}{g_{YM}^2} \sim n$ to infinity. From the string theory embedding we can we rescale $v \sim n$ and keep g_s constant. In the $SU(2)$ gauge theory this implies that the radius of the magnetic bag $R_m \sim \frac{n}{v}$ remains constant while making the large n limit.

An interesting generalization of the correspondence between moduli spaces of gauge theories and magnetic monopoles has been found in [45]. Here it is shown that the Coulomb branch of $\mathcal{N} = 2$ $SU(n)$ Yang-Mills theory defined on $\mathbf{R}^3 \times \mathbf{S}^1$ is isomorphic to the moduli space of n periodic monopoles (to be defined soon). Now we give some details about this correspondence. The gauge theory is compactified on a circle of radius L. When the radius shrinks to zero we obtain $\mathcal{N} = 4$ $SU(n)$ Yang-Mills theory defined on $\mathbf{R}^3$ whose moduli space has been discussed previously. For general radius the moduli space is always $4(n-1)$ real dimensional.[11] The periodic monopoles are solutions of the Bogomol'nyi equations on $\mathbf{R}^2 \times \mathbf{S}^1$ where the radius of the circle is $\widetilde{L} = \frac{1}{L}$. We can think of them as solutions of the Bogomol'nyi equations in $\mathbf{R}^3$ and periodic in the x_3 direction with period $2\pi\widetilde{L}$. Periodic monopoles have been further investigated in [46] and [47]. The first important feature of periodic monopoles is that they are not finite in energy. This has to do with the divergence of a charge in two dimensions at spatial infinity. Another feature, that is related to the previous one, is that the Higgs field cannot approach a constant value at $|s| \to \infty$ but it diverges logarithmically. (This is also related to the logarithmical running of the gauge coupling at high energy, at high energy the theory is four dimensional.) The boundary condition is thus:

$$\phi \sim \frac{n}{2\pi\widetilde{L}} \log\left(|s|\,\Lambda\right) + \ldots \tag{7.1}$$

where $s = x_1 + ix_2$, ϕ is the norm of the Higgs field and the ellipses stand for subleading corrections. For dimensional reasons we have to introduce a scale Λ in Eq. (7.1). From now on we call Λ dynamical scale. The coefficient $\frac{n}{2\pi\widetilde{L}}$ is determined by the Bogomol'nyi equations and is equal to the linear charge density of the n

[11] In the decompactification limit ($R \to \infty$) the theory becomes a $\mathcal{N} = 2$ $SU(n)$ Yang-Mills theory in $d = 3+1$. The Coulomb branch of this theory, with all quantum corrections, has been determined by the Seiberg-Witten solution. In the decompactification limit there is a discontinuity in the dimension of the moduli space which is $2(n-1)$ real dimensional.

periodic monopoles.

We now discuss what we expect to happen to the n periodic multi-monopole in the $\widetilde{L} \to 0$ limit that correspond to decompactification limit of the related gauge theory. Dealing with periodic multi-monopoles there is an important dimensionless quantity to consider $2\pi\widetilde{L}\Lambda$, i.e. the ratio between the period $2\pi\widetilde{L}$ and the inverse of the dynamical scale.

First we discuss the spherical multi-monopole. If $2\pi\widetilde{L}\Lambda \gg 1$ we have a chain of spherical magnetic bags and the distance between them is much larger than their radius that is

$$R_{\mathrm{m}} = \frac{n}{v} \simeq \frac{2\pi\widetilde{L}}{\log\left(2\pi\widetilde{L}\Lambda\right)} \ll 2\pi\widetilde{L} \ . \tag{7.2}$$

As $2\pi\widetilde{L}\Lambda$ is decreased the radius of the magnetic bags will increase until they touch and they will form a unique surface separating the internal phase from the external one. When $2\pi\widetilde{L}\Lambda \ll 1$ it is easy to imagine that the resulting configuration will be that of a *magnetic tube*. To determine the radius R_{t} of the tube we cannot use a minimization energy approach as we have used for the magnetic bag since the energy is now infinite. But we can solve the problem combining the abelian Bogomol'nyi equations $\vec{\nabla}\varphi = \vec{\nabla}\phi$ and the boundary condition (7.1) (here we are using the same conventions of Section 4, φ is the magnetic scalar potential and ϕ is the norm of the Higgs field). The norm of the Higgs field is completely determined by the following conditions: it must vanish at the radius $|s| = R_{\mathrm{t}}$, it must be an harmonic function and at infinity must approach (7.1). Thus we obtain

$$\phi = \frac{n}{2\pi\widetilde{L}} \log\left(\left|\frac{s}{R_{\mathrm{t}}}\right|\right) \ , \tag{7.3}$$

and the radius of the tube is the inverse of the dynamical scale $R_{\mathrm{t}} = \frac{1}{\Lambda}$. The 't Hooft scaling of the coupling constant, that correspond to $v \sim n$, does not change the parameter $2\pi\widetilde{L}\Lambda$ and neither the size or the shape of the surface.

It is interesting to discuss another configuration, that of axial symmetric periodic multi-monopoles. We point the ax of the axial symmetric multi-monopoles in the x_1 direction. We thus have a chain of magnetic discs located at $(0, 0, k \cdot 2\pi\widetilde{R})$ that lie in the (x_2, x_3) plane. If $2\pi\widetilde{L}\Lambda \gg 1$ we have a chain of magnetic discs and the distance

between them is much larger than their radius

$$R_{\rm d} = \frac{\pi n}{2v} \simeq \frac{4\pi \widetilde{L}}{\pi \log\left(2\pi\widetilde{L}\Lambda\right)} \ll 2\pi\widetilde{L}\ . \tag{7.4}$$

When $2\pi\widetilde{L}\Lambda \ll 1$ the configuration becomes a *magnetic strip* described by $x_1 = 0$, $-R_{\rm s} \leq x_2 \leq R_{\rm s}$ and $-\infty \leq x_3 \leq \infty$. The field ϕ must be an harmonic function that vanishes on the strip and has the boundary conditions (7.1). The solution is

$$\phi = \frac{n}{2\pi\widetilde{L}}\, \mathrm{Re}\cosh^{-1}\left(\frac{s}{R_{\rm s}}\right) \tag{7.5}$$

and the thickness of the strip is twice the radius of the tube $R_{\rm s} = \frac{2}{\Lambda}$. The same result (7.5) has been obtained in [46][12].

8 Multi-monopoles and Cosmology

If we want to make a phenomenological discussion about magnetic monopole, we inevitably have to confront with cosmology. In fact magnetic monopoles, if they exist, arise as stable solitons of the grand unification symmetry breaking. If their masses are of order 10^{16} GeV, the only way they can be produced is in the cosmological contest, when the temperature of the universe was of order of the GUT scale. In this section we want to address the following question:

- Is it possible to have multi-monopole formation in the cosmological context?

The answer is yes if we can play with some parameters of the GUT Higgs potential. The only two parameters that enter in the game are $V(0)$, the value of the Higgs potential in zero, and $V''(v)$ that corresponds to the mass of the neutral Higgs boson. We will also see that our mechanism for the production of multi-monopoles brings also another consequence: a reduction of the total number of monopoles plus anti-monopoles. We thus can address another question:

- Is it possible to solve the monopole cosmological problem?

[12]Here $n = 1$ and the distance $\widetilde{R}$ is sent to zero. The result is still a magnetic strip because in a chain of periodic single monopoles they have all the same $U(1)$ phase factor. When two monopoles with the same $U(1)$ phase collide they create an axial symmetric multi-monopole.

The answer is again yes, but now we have to make an extreme choice of the parameter $V''(v)$.

In the Subsection 8.1 we briefly review the ordinary theory of the monopole production during the GUT phase transition. In Subsection 8.2 we provide a possible mechanism for the formation of multi-monopoles and finally in Subsection 8.3 we discuss the possible solution of the cosmological monopole problem.

8.1 Monopoles and Cosmology

Now we briefly recall the cosmological monopole production (see [58] for a review). When the universe cooled below the critical temperature of the GUT phase transition T_c, the scalar field ϕ condensed in various domains of length ξ (see Figure 18). The

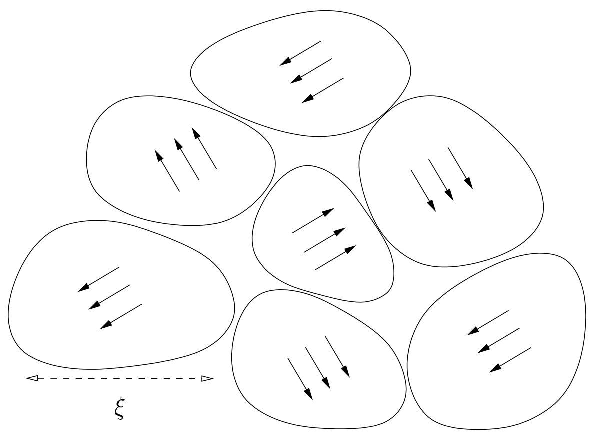

Figure 18: The Kibble mechanism. In any cosmological phase transition the order parameter is correlated in domains of finite length ξ. The length ξ must be finite since it is bounded from below by the horizon length d_H.

finiteness of the length ξ is the crucial point for the existence of topological defects [48]. Even if the correlation length becomes infinite at the critical temperature, the length ξ is always bounded from below by the horizon scale d_H. At the intersection of the various domains there is a probability p of finding a topological defect and $p \sim 1/10$ in the case of the monopole. This implies that we can neglect the production of multi-monopoles at this stage. The outcome of the phase transition is presented in Figure 19. We have a distribution of single monopoles of density $d^{-3} \sim p\xi^{-3}$.

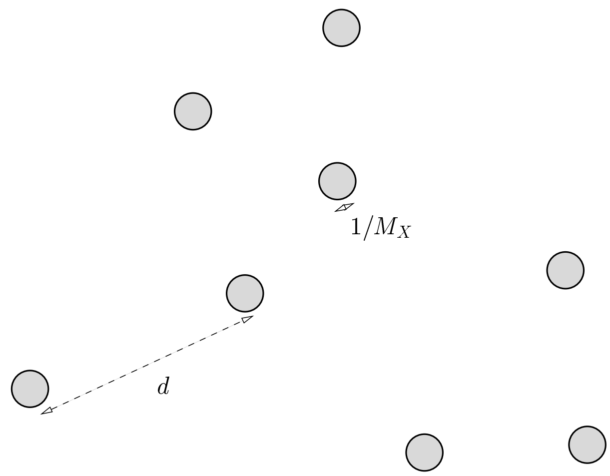

Figure 19: At the intersection of the various domains in Figure 18, there is a probability p of finding a topological defect. The outcome of the phase transition is thus a homogeneous distribution of monopoles and anti-monopoles with size $1/M_X$ and mean distance d, where $d^{-3} \sim p\xi^{-3}$.

Let's take a look at the various order of magnitudes in the problem. The mass of the heavy GUT bosons is $M_X \sim 10^{15}\,\mathrm{GeV}$ and is almost the same as the critical temperature T_c. At this temperature the horizon scale is $d_H \sim (10^{12}\,\mathrm{GeV})^{-1}$. If we take the extreme case in which the Kibble bound is saturated, we thus have a distribution of monopoles and anti-monopoles of typical distance $d \sim (10^{12}\,\mathrm{GeV})^{-1}$ and radius $R_\mathrm{m} \sim {M_X}^{-1} \sim (10^{15}\,\mathrm{GeV})^{-1}$. Generally the mass of the GUT Higgs boson is considered of the same order of the X boson. Thus our configuration satisfy the following conditions:

$$R_\mathrm{m} \sim \frac{1}{M_X} \sim \frac{1}{M_H} \ll d \ . \tag{8.1}$$

In this regime we can treat the system as a neutral plasma of monopoles and anti-monopoles whose interaction is only due to the magnetic field. At this stage the only physical effect that can happen is the annihilation between monopoles and anti-monopoles. Since the system is neutral, there is only a small drift force that cause the attraction between them. The calculations in [49, 50] show that this process, when we take into account also the expansion of the universe, is essentially negligible. The predicted density of monopoles and anti-monopoles is many order of magnitudes bigger than the upper bound posed by the observations. This is the so called cosmological monopole problem, an enormous discrepancy between the prediction of the GUT models and the observational bounds.

A lot of possible solutions have been proposed to this problem. One is in the context of inflation [51, 52]. If the GUT phase transition is before or during inflation, the density of monopoles and anti-monopoles can be enormously diluted by the exponential expansion of the universe. Another possible solution is that the universe undergoes a intermediate phase transition where the electromagnetic $U(1)$ is in the Higgs phase [53]. In this phase monopoles and anti-monopoles are confined by flux tubes and the annihilation process is enhanced. More recent speculations on the subject are considered in Refs. [54].

8.2 A mechanism for the formation of multi-monopoles

As we previously mentioned, the production of multi-monopoles can be considered negligible in the usual scenario. If we want to explore the possibility of the production of multi-monopoles, we need to change the potential for the GUT symmetry breaking. In particular there are two parameters that play a fundamental role: the zero energy density $V(0) = \varepsilon_0$ and the Higgs boson mass $M_H = V''(v)$ (see Figure 20).

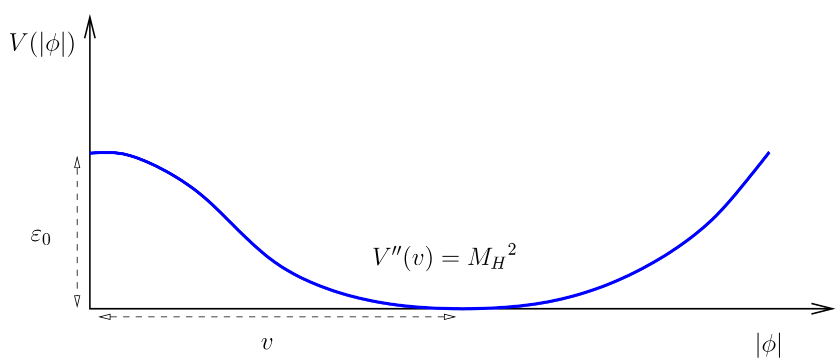

Figure 20: A potential that can lead to the production of multi-monopoles.

First of all we evaluate the spectrum of multi-monopoles using the results of Section 3. We just have to plot, like in Figure 4, the BPS mass and the MIT bag mass

$$M_{\rm BPS}(n) = \frac{4\pi v}{e}\, n \ , \qquad M_{\rm MIT}(n) = \frac{2^{1/4} 7\pi}{3} \frac{{\varepsilon_0}^{1/4}}{e^{3/2}}\, n^{2/3} \ , \tag{8.2}$$

and they intersect at

$$n^* \sim 2e \frac{v^2}{\sqrt{\varepsilon_0}} \ . \tag{8.3}$$

Far away from the transition between the two regimes, we can approximate the mass as $M_{\rm m} = \max(M_{\rm BPS}, M_{\rm MIT})$. Taking ε_0 sufficiently smaller than v^4, there is a long BPS region between $1 < n < n^*$, then a small transition between the two regimes and finally the MIT bag regime $n > n^*$. Up to the value n^* we can have marginally stable multi-monopoles, above that number the multi-monopoles will be unstable to decay into multi-monopoles of magnetic charge lower than n^*. From (8.3) we see that we the smaller ε_0 is, the more the region of marginal stability is wide.

If we want to change the evolution of monopoles after the phase transition, we need to change something in the inequalities (8.1). What we want is a regime in which the following inequalities are satisfied:

$$R_{\rm m} \sim \frac{1}{M_X} \ll d \ll \frac{1}{M_H} \ , \tag{8.4}$$

and the situation is described in Figure 21. Consider a sphere of radius $1/M_H$ that

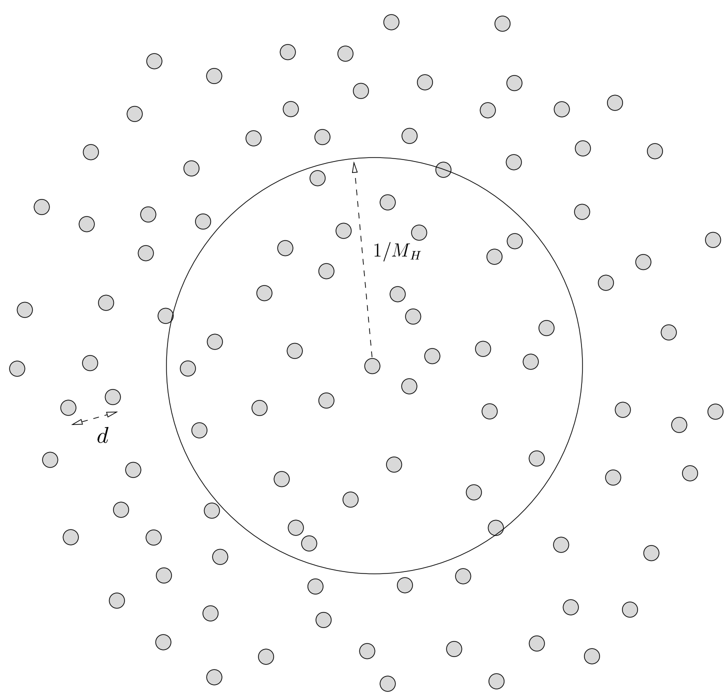

Figure 21: Inside the sphere of radius $1/M_H$ the correct approximation is that of a plasma of BPS monopoles.

is in the regime (8.4), the radius is much larger than the mean distance d. Inside this sphere we cannot approximate the monopoles as a plasma of particles interacting

only with the magnetic field. The correct approximation is that of a plasma of BPS monopoles, where the force due to the exchange of the Higgs boson gives a fundamental contribution to the dynamics. In particular the force between two BPS monopoles is zero while the force between a monopole and an anti-monopole is doubly attractive [23]. So the physics inside a sphere of radius $1/M_X$ is very similar that of a system of particles with gravitational interaction, we have only $1/r^2$ attractive forces and no repulsive forces. It is known from the theory of structure formation, that the expansion of the universe cannot stop the gravitational collapse but only change its dependence on time from exponential to polynomial.

In the theory of gravitational instability [66] there is a particular quantity to take care of: the Jeans length λ_J. If the scale of the density fluctuation is greater than λ_J we can have a collapse, otherwise the fluctuation will continue to oscillate without growing. The collapse of the sphere in Figure 21 can happen only if $1/M_H$ is greater than the Jeans length. A crude estimate of λ_J can be obtained as follows. We take a sphere of radius λ and we consider a particle at the edge of the sphere. The potential energy of the particle is $\sim N/\lambda$ where $N \sim (\lambda/d)^3$ is the number of particles in the sphere. The Jeans length is the one at which the potential energy becomes comparable to the kinetic energy. Above this scale the potential energy dominates over the dissipation and the clustering can happen. Taking $d \sim (10^{12}\,\mathrm{GeV})^{-1}$ and the kinetic energy $10^{15}\,\mathrm{GeV}$ we obtain $\lambda_J \sim (10^9\,\mathrm{GeV})^{-1}$. We thus have the constraint $M_H < 10^9\,\mathrm{GeV}$ if we want the collapse to take place.

At this point we need to say something about the statistical behavior of monopole and anti-monopoles. We call n_+ and n_- the number of monopoles and antimonopoles inside a sphere of radius $1/M_H$. The number of particles is

$$n = n_+ + n_- \sim \frac{1}{(M_H d)^3} \ . \tag{8.5}$$

The total charge is $\delta n = n_+ - n_-$. If we consider all the possible spheres of radius $1/M_H$, δn is a stochastic variable with zero mean $\langle \delta n \rangle = 0$. The physical quantity we are interested on is the variance $\sqrt{\langle \delta n \rangle}$ that, from now on, we denote for simplicity δn. Suppose for a moment that the n particles inside the sphere are independent stochastic variables. Every particle can assume the value $+1$ (monopole) or -1 (anti-monopole) with equal probability. This would give a variance $\delta n \sim \sqrt{n}$. This naive expectation is wrong since the particles are strongly correlated. The total magnetic

charge can in fact be expressed as an integral over the surface of the sphere [55]

$$\delta n = \frac{1}{8\pi} \int ds^{ij} \frac{\epsilon_{abc}\phi^a \partial_i \phi^b \partial_j \phi^c}{|\phi|^3} \ . \tag{8.6}$$

This implies that the magnetic charge fluctuation is a surface effect and not a volume effect. The variance is thus the square root of the surface area [63]

$$\delta n \sim \frac{1}{M_H d} \ . \tag{8.7}$$

The process we have just described has brought another (in principle unrequested) consequence: a reduction of the total number of monopoles and anti-monopoles. Before the collapse we had a number $1/(M_H d)^3$ of monopoles and anti-monopoles. At the end of the collapse we have a big multi-monopole of charge $1/(M_H d)$. The total number of particles has been reduced by a factor $1/(M_H d)^2$.

8.3 The cosmological monopole problem

It is interesting to ask if this process could solve the cosmological monopole problem. In principle it could work, we just have to take the mass of the GUT Higgs small enough. If we take $d \sim (10^{12}\,\mathrm{GeV})^{-1}$ and $M_H \sim 10^3\,\mathrm{GeV}$ we have a suppression of 10^{18}, that is enough to solve the monopole problem. But now the monopole problem is translated into a hierarchy problem, the mass of the GUT Higgs boson is considerably lighter than the GUT scale. This is not so bad if this hierarchy has same relation with the electroweak hierarchy. It is believed that at $10^3\,\mathrm{GeV}$ some new physics will be discovered. This new physics (technicolor, supersymmetry or something that is still unknown) should explain why the electroweak scale is so small compared to the GUT scale.

To summarize we have the following scenario. We have a GUT scale at $10^{15}\,\mathrm{GeV}$ and some "protection" at 1 TeV that solves the electroweak hierarchy problem. Now suppose that the GUT Higgs boson is essentially massless at the GUT scale and acquires mass only at 1 TeV. After the GUT phase transition a certain distribution of monopoles and anti-monopoles is created. The subsequent evolution of these particles is usually described by a neutral plasma of charged particles. This approximation does not work in our scenario. The GUT Higgs boson gives an essential contribution and, inside a sphere of radius $1/M_H$, we can approximate our system as a plasma of

almost-BPS monopoles. The physics of this system is very similar to that of a plasma of gravitational interacting particles and the collapse, even if it is slowed due to the expansion of the universe, is unavoidable. Our model predicts a reduction of the number of monopoles and anti-monopoles of order $1/(M_H d)^2 \sim 10^{-18}$. This number is big enough to solve the monopole problem and small enough to leave us the chance to observe magnetic monopoles.

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